RELATION BETWEEN THE DIMENSIONS OF THE RING GENERATED BY A VECTOR BUNDLE OF DEGREE ZERO ON AN ELLIPTIC CURVE AND A TORSOR TRIVIALIZING THIS BUNDLE

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1. Introduction and Notations

Let X be a complete, connected, reduced scheme over a perfect field k. We define $\mathrm{Vect}(X)$ to be the set of isomorphism classes $[V]$ of vector bundles V on X. We can define an addition and a multiplication on $\mathrm{Vect}(X)$:

$$[V]+[V']=[V\oplus V']$$
$$[V]\cdot[V']=[V\otimes V'].$$

The (naive) Grothendieck ring $K(X)$ (see [1]) is the ring associated to the additive monoid $\mathrm{Vect}(X)$, that means

$$K(X)=\frac{\mathbb{Z}[\mathrm{Vect}(X)]}{H},$$

where H is the subgroup of $\mathbb{Z}[\mathrm{Vect}(X)]$ generated by all elements of the form $[V\oplus V']-[V]-[V']$.
The indecomposable vector bundles on X form a free basis of $K(X)$. Since $\mathrm{H}^0(X,\mathrm{End}(V))$ is finite dimensional, the Krull-Schmidt theorem ([3]) holds on X. This means that a decomposition of a vector bundle in indecomposable components exists and is unique up to isomophism.

We want to generalize a theorem of M. Nori on finite vector bundles. A vector bundle V on X is called finite, if the collection $S(V)$ of all indecomposable components of $V^{\otimes n}$ for all integers $n\in\mathbb{Z}$ is finite.
In the following, we denote by R(V) the $\mathbb{Q}$-subalgebra of $K(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ generated by the set $S(V)$. Thus a vector bundle V is finite if and only if the ring $R(V)$ is of Krull dimension zero.

In [1], Nori proves the following theorem:
For every finite vector bundle V on X there exists a finite group scheme G and a principal G-bundle $\pi: P\to X$, such that π^*V is trivial on P. In particular, the equality

$$\dim R(V)=\dim G\,(=0)$$

holds.
As every vector bundle V on X of rank r trivializes on its associated principal $\mathrm{GL}(r)$-bundle, we can look for a group scheme G of smallest dimension and a principal G-bundle on which the pullback of the vector bundle V is trivial. We might also compare the dimension of the group scheme to dim $R(V)$.
In this article we consider the family of vector bundles of degree zero on an elliptic curve. We will prove in propositions 2 and 3 that they trivialize on a principal G-bundle with G a group scheme of smallest dimension one.
As in the situation of Nori's theorem, this dimension turns out to be equal to the dimension of the ring $R(V)$.

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2. Dimension relation for vector bundles of degree zero on an elliptic curve

Let X be an elliptic curve over an algebraically closed field k of characteristic zero. We consider vector bundles of degree zero on X which can be classified according to Atiyah (see [2]). By $\mathcal{E}(r,0)$ we denote the set of indecomposable vector bundles of rank r and degree zero.

Theorem 1. *(Atiyah [2])*

1. *There exists a vector bundle $F_r \in \mathcal{E}(r,0)$, unique up to isomorphism, with $\Gamma(X,F_r) \neq 0$.*
   Moreover we have an exact sequence
   $$0 \;\to\; \mathcal{O}_X \;\to\; F_r \;\to\; F_{r-1} \;\to\; 0.$$
2. *Let $E \in \mathcal{E}(r,0)$, then $E \cong L \otimes F_r$ where L is a line bundle of degree zero, unique up to isomorphism (and such that $L^r \cong \det E$.)*

Proposition 2.

i) *The $\mathbb{Q}$-subalgebra $R(F_r)$ of $K(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ generated by $S(F_r)$ is $\mathbb{Q}[x]$, where $x = [F_2]$, if r is even, and $x = [F_3]$, if r is odd. In particular, $R(F_r)$ is of Krull dimension zero.*
ii) *There exists a principal $\mathbb{G}_a$-bundle $\pi : P \to X$ such that $\pi^*(F_r)$ is trivial for all $r \geq 2$.*

Remark: As in Nori's case we have a correspondence of dimensions

$$dim\ R(F_r) = dim\ \mathbb{G}_a = 1.$$

Proof:
As proved by Atiyah in [2], the vector bundles F_r are self-dual and fulfill the formula

$$F_r \otimes F_s = F_{r-s+1} \oplus F_{r-s+3} \oplus \cdots \oplus F_{(r-s)+(2s-1)} \text{ for } s \leq r.$$

For even r, it follows by induction that there exist integers $a_i(n)$ such that

$$F_r^{\otimes n} = a_2(n)F_2 \oplus a_4(n)F_4 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}$$

for odd $n \geq 3$, and

$$F_r^{\otimes n} = a_1(n)\mathcal{O}_X \oplus a_3(n)F_3 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}$$

for even $n \geq 2$.
Therefore we obtain

$$S(F_r) = \{F_i \,|\, i = 1, 2, 3, \ldots\}\,, \text{ if } r \text{ even },$$

and $S(F_r)$ generates the subring $\mathbb{Q}[F_2]$ of $K(X) \otimes \mathbb{Q}$, because inductively we can write every vector bundle F_i as $p(F_2)$ for some polynomial $p \in \mathbb{Z}[x]$.

For odd r, Atiyah's multiplication formula gives

$$F_r^{\otimes n} = a_1(n)\mathcal{O}_X \oplus a_3(n)F_3 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}$$

for all $n \geq 2$. It follows that

$$S(F_r) = \{F_i \,|\, i \text{ odd }\}\,, \text{ if } r \text{ odd }.$$

For odd r, the set $S(F_r)$ generates the ring $R(F_r) = \mathbb{Q}[F_3]$, as for odd i each F_i is $p(F_3)$ for a polynomial $p \in \mathbb{Z}[x]$.

The vector bundle F_2 is an element of $H^1(X, \mathrm{GL}(2, \mathcal{O}))$. Because of the exact sequence

$$0 \;\to\; \mathcal{O}_X \;\to\; F_2 \;\to\; \mathcal{O}_X \;\to\; 0,$$

F_2 is even an element of $H^1(X, \mathbb{G}_a)$. Here we embed $\mathbb{G}_a$ into $\mathrm{GL}(2, \mathcal{O})$ via $u \to \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Hence F_2 trivializes on a principal $\mathbb{G}_a$-bundle. As $F_r = S^{r-1}F_2\,, r \geq 3$, each F_r trivializes on the same principal $\mathbb{G}_a$-bundle as F_2.

As the classes $[F_r]$ are not torsion elements in $H^1(X, \mathrm{GL}(2, \mathcal{O}))$, none of the bundles F_r can trivialize on a principal G-bundle with G a finite group scheme. $\square$

Remark: In the given examples of vector bundles E there was so far not only a correspondence of the dimensions of the group scheme

and the ring $R(E)$. The algebra $R(E)$ was also the Hopf algebra corresponding to the group scheme. The following proposition shows that this is not true in general.

Proposition 3. *Let $E \cong L \otimes F_r \in \mathcal{E}(r,0)$ (see theorem 1).*

1. *If L is not torsion, the ring $R(E)$ is isomorphic to $\mathbb{Q}[x,x^{-1}] \otimes \mathbb{Q}[y]$ and E trivializes on a principal $\mathbb{G}_m \times \mathbb{G}_a$-bundle.*
2. *If L is torsion, let $n \in \mathbb{N}$, $n \geq 1$, be the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$. If n and r are both even, the ring $R(E)$ is isomorphic to*

$$\mathbb{Q}[x]/ < x^{n/2} - 1 > \otimes \mathbb{Q}[y]$$

and E trivializes on a principal $\mu_n \times \mathbb{G}_a$-bundle. There is no principal $\mu_{n/2} \times \mathbb{G}_a$-bundle where E is trivial.
If n and r are not both even, the ring $R(E)$ is isomorphic to

$$\mathbb{Q}[x]/ < x^{n} - 1 > \otimes \mathbb{Q}[y]$$

and E trivializes on a principal $\mu_n \times \mathbb{G}_a$-bundle.

Proof: Let $E \in \mathcal{E}(r,0)$ with $\Gamma(X,E) = 0$. (If $\Gamma(X,E) \neq 0$, then $E \cong F_r$. This case was already dealt with in proposition 2.)

First we consider the case that L is not torsion.
We must distinguish between odd and even r.
For odd r, Atiyah's multiplication formula (see proof of proposition 4) gives the following result:
For $m \in \mathbb{N}$, $m \geq 2$, the tensor power $E^{\otimes m} \cong L^{\otimes m} \otimes F_r^{\otimes m}$ has the indecomposable components $L^{\otimes m} \otimes \mathcal{O}_X$, $L^{\otimes m} \otimes F_3, \dots, L^{\otimes m} \otimes F_{(r-1)m+1}$, the tensor power $E^{\otimes -m} \cong L^{\otimes -m} \otimes F_r^{\otimes m}$ has the indecomposable components $L^{\otimes -m} \otimes \mathcal{O}_X$, $L^{\otimes -m} \otimes F_3, \dots, L^{\otimes -m} \otimes F_{(r-1)m+1}$.
Thus we obtain

$$S(E) = \left\{ \begin{array}{l} \mathcal{O}_X, L \otimes F_r, L^{-1} \otimes F_r, \\ L^{\otimes \pm i} \otimes F_3, L^{\otimes \pm i} \otimes F_5, \dots, L^{\otimes \pm i} \otimes F_{(r-1)i+1}, \text{ i } \in \mathbb{N} \end{array} \right\}.$$

The algebra $R(E)$ which is generated by $S(E)$ is the subalgebra of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by L, L^{-1} andF_3, thus

$$R(E) = \mathbb{Q}[L, L^{-1}] \otimes_{\mathbb{Z}} \mathbb{Q}[F_3].$$

For even r, a similar computation gives that

$$S(E) = \left\{ \begin{array}{l} \mathcal{O}_X, L \otimes F_r, L^{-1} \otimes F_r, \\ L^{\otimes \pm 2i}, L^{\otimes \pm 2i} \otimes F_3, \dots, L^{\otimes \pm 2i} \otimes F_{(r-1)2i+1}, \text{ i } \in \mathbb{N} \\ L^{\otimes \pm (2i+1)} \otimes F_2, L^{\otimes \pm (2i+1)} \otimes F_4, \dots, \\ \quad L^{\otimes \pm (2i+1)} \otimes F_{(r-1)(2i+1)+1}, \text{ i } \in \mathbb{N} \end{array} \right\}.$$

The ring $R(E)$, generated by $S(E)$, is the subring of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is generated by the elements $L^{\otimes 2}$, $L^{\otimes -2}$, $L^{-1} \otimes F_2$, therefore

$$R(E) = \mathbb{Q}[L^{\otimes 2}, L^{\otimes -2}] \otimes_{\mathbb{Z}} \mathbb{Q}[L^{-1} \otimes F_2].$$

If L is not a torsion bundle, it is clear that L trivializes on a principal $\mathbb{G}_m$-bundle P_L. The vector bundle $E \cong L \otimes F_2$ trivializes on the $\mathbb{G}_m \times \mathbb{G}_a$-bundle $P_L \times_X P$, where P is the principal $\mathbb{G}_a$-bundle from proposition 2, where F_2 and hence all the F_r trivialize.

Let now L be torsion and $n \in \mathbb{N}$, $n \geq 2$, the minimal number with $L^{\otimes n} \cong \mathcal{O}_X$. As the F_r are selfdual and $L^{\otimes n-1} = L^{-1}$, it suffices to consider positive tensor powers.
Again we compute the tensor powers using Atiyah's formula to find the indecomposable components.

If r is even and n is odd, the set $S(E)$ contains the following bundles:

$$S(E) = \{\mathcal{O}_X, L^{\otimes i} \otimes F_j \,|\, i = 0, 1, \ldots, n-1,\, j \in \mathbb{N}\}.$$

With the help of the multiplication formula for F_2 it is easy to show that all elements of $S(E)$ can be generated by L and F_2. In additon, the relation $L^{\otimes n} \cong \mathcal{O}_X$ holds. Hence we obtain

$$R(E) = \frac{\mathbb{Q}[L]}{< L^{\otimes n} - 1 >} \otimes_{\mathbb{Z}} \mathbb{Q}[F_2].$$

If r is odd and n is even or odd, the result is

$$S(E) = \{L^{\otimes i} \otimes F_j \,|\, i = 0, 1, \ldots, n-1,\, j \in \mathbb{N} \text{ odd}\}.$$

The bundles L and F_3 are in $S(E)$ and generate all elements of $S(E)$. Because of the relation $L^{\otimes n} \cong \mathcal{O}_X$, the algebra $R(E)$ is

$$R(E) = \frac{\mathbb{Q}[L]}{< L^{\otimes n} - 1 >} \otimes_{\mathbb{Z}} \mathbb{Q}[F_3].$$

If r and n are both even

$$S(E) = \{L^{\otimes 2i} \otimes F_{2j-1}, L^{\otimes 2i+1} \otimes F_{2j} \,|\, i = 0, 1, \ldots, n/2,\, j \in \mathbb{N}\}.$$

The algebra R(E) is generated by $L^{\otimes 2}$ and $L \otimes F_2$. The generators are subject to the relation $L^{\otimes n} \cong \mathcal{O}_X$, thus

$$R(E) = \frac{\mathbb{Q}[L^{\otimes 2}]}{< (L^{\otimes 2})^{\otimes m} - 1 >} \otimes \mathbb{Q}[L \otimes F_2],$$

where $m = n/2$.

Recall that $n \geq 2$ is the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$. Thus the bundle L trivializes on a μ_n-bundle P_L and not on a μ_m-torsor for $m < n$.
The bundle $E \cong L \otimes F_r$ then trivializes on the $\mu_n \times \mathbb{G}_a$-bundle $P_L \times_X P$,

where P is again the principal $\mathbb{G}_a$-bundle from proposition 2. We will now show that the bundle E does not trivialize on a $\mu_{n/2} \times \mathbb{G}_a$-bundle: If $E \cong L \otimes F_r$ trivializes on $Q \times_X P$, where Q is a μ_m-torsor and P a $\mathbb{G}_a$-torsor, then $\det(L \otimes F_r) = L$ is the identity element in the group $\mathrm{Pic}(Q \times_X P)$. But one has $\mathrm{Pic}(Q \times_X P) = \mathrm{Pic}(Q)$ by homotopy invariance. Thus L must trivialize on the μ_m-torsor Q, which is impossible for $m < n$. $\square$

Remark: The correspondence between the dimension of the "minimal" group scheme and the dimension of the ring $R(E)$ also occurs in the case of vector bundles on the projective line, as one easily sees.
Let X be the complex projective line $\mathbb{P}^1$ and $E := \mathcal{O}(a)$ a line bundle.
If $a = 0$ we have $S(E) = \{\mathcal{O}\}$ and $R(E) = Q$.
We define the group scheme G to be $G = \mathrm{Spec}\ \mathbb{Q}$ and the trivializing torsor is simply $\mathbb{P}^1$.
If $a \neq 0$ we can easily compute that $S(E) = \{\mathcal{O}(\lambda \cdot a) | \lambda \in \mathbb{Z}\}$ and $R(E) = \mathbb{Q}[x, x^{-1}]$. We define the group scheme to be $G = \mathbb{G}_m = \mathrm{Spec}\ \mathbb{Q}[x, x^{-1}]$.
The given line bundle E trivializes on a principal $\mathbb{G}_m$-bundle P_a, which depends on a.
Thus we get the correspondence of $\dim R(E)$ and $\dim G$ in the case of a line bundle on $\mathbb{P}^1$. This computation can easily be generalized to the case of vector bundles of higher rank. We illustrate this for bundles of rank two.
Let now E be a vector bundle of rank 2 on $\mathbb{P}^1$, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$.
The case $(a, b) = (0, 0)$ is trivial. We can see at once that $S(E) = \{\mathcal{O}\}$ and therefore $R(E) = \mathbb{Q}$.
The vector bundle E trivializes on the principal $\mathrm{Spec}\ \mathbb{Q}$ - bundle $\mathbb{P}^1$.
If $(a, b) \neq (0, 0)$ the computation gives that $S(\mathcal{O}(a) \oplus \mathcal{O}(b)) = S(\mathcal{O}(c))$, where $c = (a, b)$ (with $(a, 0) = a$ and $(0, b) = b$) and therefore $R(E) = \mathbb{Q}[x, x^{-1}]$. E trivializes on the principal $\mathbb{G}_m$-bundle P_c that belongs to $\mathcal{O}(c)$ as $\mathcal{O}(a) = \mathcal{O}(c)^\lambda$ and $\mathcal{O}(b) = \mathcal{O}(c)^\mu$ for appropriate integers λ and μ.

References

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